Anomalous Collisionless Damping of Alfvén Waves in Inhomogeneous Plasmas

F. WINTERBERG

Desert Research Institute and Department of Physics, University of Nevada, Reno, Nevada

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It is shown that ALFVÉN waves in a collisionless plasma, permeated by an inhomogeneous magnetic field, are affected by a strong anomalous damping. This anomalous collisionless damping seems to correspond to collision-free LANDAU damping in homogeneous magnetic fields in a similar way as the anomalous viscous and OHMIC damping of ALFVÉN waves propagating in inhomogeneous magnetic fields [1] corresponds to the ordinary collisional viscous and OHMIC damping of ALFVÉN waves propagating in homogeneous magnetic fields.

It is found that the predicted collision-free anomalous damping will obey a law of the form $\exp\{-(t/\tau)^2\}$ where the damping time τ depends on the wave-length, the temperature of the plasma, and the magnetic field gradient.

Starting from a simplified BOLTZMANN–VLASOV equation, valid for low frequency plasma waves [2], a LAPLACE transform is carried out with regard to the time dependence and an integral equation is derived for the mean plasma velocity. By the faltung theorem this integral equation can be FOURIER transformed into a differential equation for which a W.K.B. solution can be obtained. The inverse transforms can be carried out by making a simple analytic approximation for the W.K.B. solution which is valid over a large range of values.

The predicted anomalous damping of ALFVÉN waves is so strong that it may be useful for the purpose of heating a plasma up to high temperatures within a short time.

1. Introduction

Let us consider a plasma in equilibrium with a magnetic field in which the field lines form closed lines within the plasma configuration. An example is the torus-like field configuration of the stellarator. Let us furthermore consider a standing ALFVÉN wave propagating along a field line located within the plasma. If the circumference of the field line is l, the standing ALFVÉN wave must obey periodicity conditions

$$v(s+l)=v(s),\quad \frac{\partial}{\partial s}v(s+l)=\frac{\partial}{\partial s}v(s), \tag{1.1}$$

in (1.1) v is the fluid velocity associated with the ALFVÉN wave and s the arc length measured along the field line.

The standing ALFVÉN wave propagating along the field line will have a frequency ω proportional to

$$\omega \propto \oint \frac{\mathrm{d}s}{u(s)}, \tag{1.2}$$

where
$$u(s)=H_0(s)/\sqrt{4\pi\varrho(s)} \tag{1.3}$$

is the ALFVÉN speed along the lines of force, in general a function of s; $\varrho(s)$ is the plasma density. $H_0(s)$ is the strength of the magnetic field which is in equilibrium with the undisturbed plasma configuration, thus obeying the magnetohydrostatic equation

$$\operatorname{curl}[\boldsymbol{H}_0\times\operatorname{curl}\boldsymbol{H}_0]=0\,. \tag{1.4}$$

Let us now consider another ALFVÉN wave propagating along a second field line. The frequency of this second wave will, in general, be different from the frequency of the first wave, since $\oint \mathrm{d}s/u(s)$ will be different for different field lines.

If both standing ALFVÉN waves propagating along different field lines are at some initial time in phase, they will move out of phase as time passes. If we consider, instead of the two ALFVÉN waves propagating along the two field lines, a whole assembly of waves propagating along many field lines; that is, if we consider the whole plasma as being excited by standing ALFVÉN waves, then all waves belonging to different field lines will get out of phase. Since the difference in frequency will be small for waves propagating along field lines with a small spatial separation, the rule will be that, first, waves propagating along field lines well separated from each other will get out of phase, then when enough time has passed, waves propagating along field lines which are very close to each other will get out of phase. As a consequence, large velocity gradients perpendicular to the direction of the magnetic field lines, will build up in time.

[1] F. WINTERBERG, Ann. Phys. N.Y. **25**, 174 [1963].

[1] F. WINTERBERG, Ann. Phys. N.Y. **29**, 259 [1964].

In a previous paper [1] it was shown that collisions, introduced by assuming a non-zero viscosity and finite conductivity, will damp out these waves rapidly.

This anomalous viscous and OHMIC damping time could become considerably shorter than the normal viscous and OHMIC damping time of an ALFVÉN wave propagating in a homogeneous magnetic field, depending on the magnitude of gradient of the magnetic field perpendicular to the field line.

In order to avoid the difficulties arising from the complicated magnetic field configurations, as for instance the torus like stellarator geometry, a much simpler inhomogeneous magnetic field has been chosen from which the same effect can be derived. Such a simple field is given by:

$$\boldsymbol{H} = \{0, H_y(x), 0\} = \{0, H_0(x), 0\}. \quad (1.5)$$

ALFVÉN waves propagating in an inhomogeneous magnetic field of the form (1.5) will get out of phase in exactly the same manner if standing waves are chosen in between two boundaries which are perpendicular to the direction of $\boldsymbol{H}$ and which are separated by a distance l.

To simplify the analysis furthermore, we are considering only ALFVÉN waves, propagating in the field given by (1.5) with a polarization of the fluid velocity vector $\boldsymbol{v}$ parallel to the z-axis. For the anomalous damping time τ (the time in which the wave energy is dissipated by a factor e^{-1}) an expression was derived of the form [1]:

$$\tau = \left[\frac{3\,\sigma}{\pi\, c^2 (1 + 4\,\pi\,\sigma\,\nu_\perp / c^2)\,(n/l)^2 (\nabla u_\perp)^2}\right]^{1/3}. \quad (1.6)$$

In eq. (1.6) σ is the electrical conductivity, $\nu_\perp$ the (perpendicular) kinematic viscosity, n the number of wave-nodes, and $\nabla u_\perp$ the gradient of the ALFVÉN speed perpendicular to the field lines.

For an ALFVÉN wave propagating in a plasma permeated by a homogeneous magnetic field, the normal damping time τ_n, resulting from viscosity and finite conductivity, is given by

$$\tau_n = \left[\frac{\pi\, c^2}{\sigma} \left(\frac{n}{l}\right)^2 \left(1 + \frac{4\,\pi\,\sigma\,\nu}{c^2}\right)\right]^{-1}, \quad (1.7)$$

from the comparison of (1.6) with (1.7) it follows that if $|\nabla u_\perp|$ is large enough then $\tau \ll \tau_n$.

The anomalous damping time τ was derived from a fluid dynamic, macroscopic model. For this reason formula (1.6) for the anomalous damping time will be valid only if τ is not smaller than the collision time. With increasing temperature the collision time becomes larger. For this reason, values of τ, computed from eq. (1.6), may become smaller than the collision time. We may, therefore, raise the question of what will happen in the limiting case of infinite collision time. Obviously, in the limiting case of infinite collision time, the dissipative terms describing non-zero viscosity and finite conductivity must be omitted. The equations are then simply the equations of a frictionless fluid of infinite conductivity. According to the fluid dynamic description it follows, then, that if time passes the waves will get out of phase completely, and as a consequence the velocity gradients perpendicular to the lines of force will increase in time without reaching a limit. This can be most clearly demonstrated from the wave equation for an ALFVÉN wave propagating in a field of the form given by (1.5), assuming that the fluid velocity vector of the wave is polarized in a direction parallel to the z-axis. Under this assumption, one easily derives as the ALFVÉN wave equation (see, for instance, ref. [1]):

$$\frac{\partial^2 v_z}{\partial t^2} - u^2(x)\,\frac{\partial^2 v_z}{\partial y^2} = 0, \quad (1.8)$$

where

$$u(x) = H_0(x)/\sqrt{4\,\pi\,\varrho(x)} \quad (1.9)$$

is the variable ALFVÉN velocity; variable with regard to the direction perpendicular to the magnetic field lines. We are looking for standing wave solutions obeying the same boundary conditions. A solution of (1.8) is then obviously

$$v_z(x, y, t) = f(x)\,\cos(\omega\, t)\,\sin(k\, y), \quad (1.10)$$

where $f(x)$ is the amplitude of the waves, in general an arbitrary function of x. Furthermore

$$k = 2\,\pi\, n/l, \qquad \omega = u\, k. \quad (1.11)$$

In (1.11) n is the number of wave nodes. If at $t = 0$ the amplitude of all waves belonging to different values of x is the same and put equal to v_0 we obtain:

$$v_z(x, y, t) = v_0 \cos(k\, u\, t)\,\sin(k\, y). \quad (1.12)$$

For the following we assume a linear dependence of the inhomogeneity in the ALFVÉN velocity

$$u(x) = a\, x = |\nabla u|\, x, \quad (1.13)$$

and obtain from (1.13)

$$v_z(x, y, t) = v_0 \cos(k\,|\nabla u|\, x\, t)\,\sin(k\, y). \quad (1.14)$$

From eq. (1.14) we can immediately derive an expression for the phase difference $\Delta\varphi$ of two waves separated by a distance Δx:

$$\Delta \varphi = k \, | \nabla u | \, t \, \Delta x \, . \quad (1.15)$$

This phase difference grows proportionally with time and is proportional to the distance Δx between the two waves and to the gradient of the magnetic field. Because the phase differences between the waves are growing with time, the velocity field as a function of x as time passes will become a more and more oscillating function of x. The wave number $\varkappa$ of this oscillating function of x is given by

$$\varkappa = k \, | \nabla u | \, t \, . \quad (1.16)$$

As a consequence, the velocity gradient perpendicular to the lines of force is also growing proportional with time. On the basis of the fluid dynamic model, our problem has a solution which is asymptotically nonanalytic and will never converge towards an equilibrium. This result, unsatisfactory from a physical point of view, has its origin in the inadequacy of the fluid dynamic picture which is inadequate for wave lengths smaller than the DEBYE-length.

A valid description of the plasma for wave lengths smaller or comparable to the DEBYE-length must employ the BOLTZMANN–VLASOV equation. One of the most important new results obtained from the BOLTZMANN–VLASOV equation is a collisionsless damping of plasma waves. This damping effect, called LANDAU damping, cannot be derived from the moment equations which are the fluid dynamic equations. LANDAU derived this damping by considering the behavior of an electronic plasma, that is, a cloud of electrons in the field of fixed, infinitely heavy ions [3]. LANDAU's assumption of infinitely heavy ions is valid for high plasma frequencies.

In a previous paper [2] the LANDAU damping of ALFVÉN waves propagating through a plasma of infinite conductivity was obtained using a simplified form of the BOLTZMANN–VLASOV equation valid for low plasma frequencies.

As there exists, for an ALFVÉN wave propagating through a homogeneous magnetic field, besides the normal collisional damping a normal LANDAU damping, we may expect that there should exist quite analogously for an ALFVÉN wave propagating through an inhomogeneous magnetic field besides the anomalous collisional damping an anomalous LANDAU damping. The purpose of this paper is to derive such an anomalous LANDAU damping and to discuss its significance with regard to plasma physics.

[3] L. LANDAU, J. Phys. USSR **10**, 25 [1946].

2. The Plasma Model

In order to describe the propagation of ALFVÉN waves in an inhomogeneous magnetic field, we use a plasma model valid for low frequencies [2]. In this model the plasma is described by a BOLTZMANN–VLASOV equation with the electromagnetic body force

$$\boldsymbol{F} = \frac{m}{\varrho} \frac{1}{c} \boldsymbol{j} \times \boldsymbol{H} \, . \quad (2.1)$$

In (2.1) m is the mass of the particles in this plasma model, being put equal to the ion mass; $\boldsymbol{j}$ is the electric current density, ϱ the plasma density and $\boldsymbol{H}$ the magnetic field.

The BOLTZMANN–VLASOV equation for the particle distribution function $F(\boldsymbol{v}, \boldsymbol{r}, \mathrm{t})$ is, therefore,

$$\frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \frac{\partial F}{\partial \boldsymbol{r}} + \frac{1}{\varrho} \left[\frac{1}{c} \boldsymbol{j} \times \boldsymbol{H} \right] \cdot \frac{\partial F}{\partial \boldsymbol{v}} = 0 \, . \quad (2.2)$$

This plasma model must be supplemented by a number of other equations.

First, an equation describing that the plasma is conducting. In the case of infinite conductivity, we may write

$$\boldsymbol{E} + (1/c) \langle \boldsymbol{v} \rangle \times \boldsymbol{H} = 0 \, . \quad (2.3)$$

In (2.3) we have

$$\langle \boldsymbol{v} \rangle = (1/\varrho) \int F(\boldsymbol{v}, \boldsymbol{r}, t) \, \boldsymbol{v} \, \mathrm{d}\boldsymbol{v} \quad (2.4)$$

which is the average or macroscopic plasma velocity.

Second, MAXWELL's equations omitting displacement currents

$$\partial \boldsymbol{H} / \partial t = -c \operatorname{curl} \boldsymbol{E} \, , \quad (2.5)$$

$$(4 \pi / c) \, \boldsymbol{j} = \operatorname{curl} \boldsymbol{H} \, , \quad (2.6)$$

$$\operatorname{div} \boldsymbol{H} = 0 \, . \quad (2.7)$$

From (2.3) together with (2.5) it follows as usual

$$\partial \boldsymbol{H} / \partial t = \operatorname{curl} \langle \boldsymbol{v} \rangle \times \boldsymbol{H} \, . \quad (2.8)$$

If we express $\boldsymbol{j}$ by $\boldsymbol{H}$ with eq. (2.6) we have

$$\frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \frac{\partial F}{\partial \boldsymbol{r}} - \frac{1}{4 \pi \varrho} [\boldsymbol{H} \times \operatorname{curl} \boldsymbol{H}] \cdot \frac{\partial F}{\partial \boldsymbol{v}} = 0 \, . \quad (2.9)$$

In the next step we linearize eq. (2.9) putting

$$F(\boldsymbol{v}, \boldsymbol{r}, t) = \varrho \left(f_0(\boldsymbol{v}, \boldsymbol{r}) + f(\boldsymbol{v}, \boldsymbol{r}, t) \right) \quad (2.10)$$

and considering f as being small of the first order; $f_0(\boldsymbol{v}, \boldsymbol{r})$ is the normalized distribution function representing the undisturbed state. We further put

$$\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{H}_0(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{r}, t) \tag{2.11}$$

and assume that $\boldsymbol{h}(\boldsymbol{r}, t)$ is a small disturbance superimposed on $\boldsymbol{H}_0(\boldsymbol{r})$. For ALFVÉN waves pressure and density do not change; we have, therefore

$$p = p_0, \qquad \varrho = \varrho_0, \tag{2.12}$$

describing the constancy of pressure and density which are put equal to the undisturbed values p_0 and ϱ_0.

We finally assume that the undisturbed plasma must obey the magnetohydrostatic equation

$$-\operatorname{grad} p = (1/4\pi)\, \boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{H}_0 . \tag{2.13}$$

From (2.9) follows for the undisturbed plasma configuration

$$\boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} (\varrho f_0) - \frac{1}{4\pi\varrho} [\boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{H}_0] \cdot \frac{\partial}{\partial \boldsymbol{v}} (\varrho f_0) = 0 \tag{2.14}$$

respectively

$$\varrho \boldsymbol{v} \cdot \frac{\partial f_0}{\partial \boldsymbol{r}} + f_0 \boldsymbol{v} \cdot \frac{\partial \varrho}{\partial \boldsymbol{r}} - \frac{1}{4\pi} [\boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{H}_0] \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = 0 . \tag{2.15}$$

Assuming that the undisturbed velocity distribution is a MAXWELLian we have

$$\partial f_0 / \partial \boldsymbol{v} = -(\varrho/p)\, f_0\, \boldsymbol{v} . \tag{2.16}$$

Inserting (2.13) and (2.16) into (2.15) it follows

$$\varrho \boldsymbol{v} \cdot \frac{\partial f_0}{\partial \boldsymbol{r}} + \frac{\varrho f_0}{p} \boldsymbol{v} \cdot \left[\frac{p}{\varrho} \operatorname{grad} \varrho - \operatorname{grad} p\right] = 0 . \tag{2.17}$$

Eq. (2.17) can be satisfied if the following conditions are fulfilled

$$(p/\varrho) \operatorname{grad} \varrho = \operatorname{grad} p , \tag{2.18}$$

and

$$\boldsymbol{v} \cdot \partial f_0 / \partial \boldsymbol{r} = 0 . \tag{2.19}$$

Condition (2.18) implies that

$$\partial p / \partial \varrho = p / \varrho , \tag{2.20}$$

which is fulfilled for an isothermal plasma, $T = \text{const.}$

If the plasma is inhomogeneous with regard to one spatial direction only, say for instance the x direction, it follows that $\partial f_0/\partial y = \partial f_0/\partial z = 0$. From (2.19) then follows as the only possible solution

$$f_0 = f_0(\boldsymbol{v}) , \tag{2.21}$$

showing that f_0 under the assumptions made can be a function of the velocity only.

For the following, we neglect the term

$$(f/\varrho_0)\, \boldsymbol{v} \cdot \partial \varrho_0 / \partial \boldsymbol{r}$$

with respect to the term $\boldsymbol{v} \cdot \partial f / \partial \boldsymbol{r}$ which means that the wavelength of a plasma disturbance in the direction of the inhomogeneity (the x-direction) shall be small compared to the characteristic length $\varrho_0/(\partial \varrho_0/\partial x)$, over which the density changes. It was shown below [eq. (1.16)] that the wavelength of the plasma disturbance in the direction of the inhomogeneity decreases with increasing time. As a consequence, the term $(f/\varrho_0)\, \boldsymbol{v} \cdot \partial \varrho_0/\partial \boldsymbol{r}$ may be neglected with increasing time.

Linearizing eq. (2.9) we thus have

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{1}{4\pi\varrho_0} [\boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{h} - \boldsymbol{h} \times \operatorname{curl} \boldsymbol{H}_0] \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = 0 . \tag{2.22}$$

Linearizing eq. (2.8) we have

$$\partial \boldsymbol{h} / \partial t = \operatorname{curl} \langle \boldsymbol{v} \rangle \times \boldsymbol{H}_0 . \tag{2.23}$$

If the undisturbed distribution function is symmetric $f_0(-\boldsymbol{v}) = f_0(\boldsymbol{v})$, which is the case for a MAXWELLian distribution function, we can write for (2.4)

$$\langle \boldsymbol{v} \rangle = \int f(\boldsymbol{v}, \boldsymbol{r}, t)\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{v} . \tag{2.24}$$

The magnetic field shall be of the form (1.5):

$$\boldsymbol{H} = \{0, \boldsymbol{H}_0(x), 0\} . \tag{2.25}$$

The velocity vector $\langle \boldsymbol{v} \rangle$ shall be polarized in the z direction. Therefore

$$\langle \boldsymbol{v} \rangle = \{0, 0, \langle v_z \rangle\} , \tag{2.26}$$

where

$$\langle v_z \rangle = \int f\, v_z\, \mathrm{d}\boldsymbol{v} . \tag{2.27}$$

From (2.23), (2.25), and (2.27) it follows that $\boldsymbol{h}$ is parallel to curl $\boldsymbol{H}_0$ thus making the term $\boldsymbol{h} \times \operatorname{curl} \boldsymbol{H}_0$ equal to zero. The result upon (2.22) is therefore

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{1}{4\pi\varrho_0} \boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{h} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = 0. \tag{2.28}$$

We differentiate eq. (2.28) with regard to time and eliminate $\boldsymbol{h}$ with the help of eq. (2.23). We obtain in this way

$$\frac{\partial^2 f}{\partial t^2} + \boldsymbol{v} \cdot \frac{\partial^2 f}{\partial \boldsymbol{r} \partial t} - \frac{1}{4\pi\varrho_0} \boldsymbol{H}_0 \times \operatorname{curl} \operatorname{curl}(\langle \boldsymbol{v} \rangle \times \boldsymbol{H}_0) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = 0 . \tag{2.29}$$

We use the fact that $\varrho_0 = \varrho_0(x)$ only and that $\boldsymbol{H}$ and $\langle \boldsymbol{v} \rangle$ are directed along the y and z axis respectively.

It is then easy to show that

$$-\frac{1}{4\pi\varrho_0}\boldsymbol{H}_0\times\operatorname{curl}\operatorname{curl}(\langle\boldsymbol{v}\rangle\times\boldsymbol{H}_0) = u^2\left\{0, 0, \frac{\partial^2\langle v_z\rangle}{\partial y^2}\right\}. \tag{2.30}$$

Using (2.30) we bring the BOLTZMANN–VLASOV equation finally into the form:

$$\frac{\partial^2 f}{\partial t^2}+\boldsymbol{v}\cdot\frac{\partial^2 f}{\partial \boldsymbol{r}\,\partial t}+u^2\frac{\partial^2\langle v_z\rangle}{\partial y^2}\frac{\partial f_0}{\partial v_z}=0. \tag{2.31}$$

3. Transformation into an Integral Equation

We shall now assume that our disturbed distribution function f does not depend upon z, that is, $\partial f/\partial z=0$. We make a FOURIER transform of f and $\langle v_z\rangle$ with regard to y, and drop afterwards the index k on the FOURIER-transformed f respectively $\langle v_z\rangle$. The result upon (2.31) is:

$$\frac{\partial^2 f}{\partial t^2}+v_x\frac{\partial^2 f}{\partial x\,\partial t}+i\,k\,v_y\frac{\partial f}{\partial t}-u^2k^2\langle v_z\rangle\frac{\partial f_0}{\partial v_z}=0. \tag{3.1}$$

We then apply upon (3.1) and on $\langle v_z\rangle$ a LAPLACE transform in time (LAPLACE transform variable p) with the result:

$$(p+i\,k\,v_y)\,p\,f_p+p\,v_x\frac{\partial f_p}{\partial x} - u^2k^2\langle v_z\rangle_p\frac{\partial f_0}{\partial v_z}=\varphi(\boldsymbol{v},x,p). \tag{3.2}$$

$\varphi(\boldsymbol{v},x,p)$ contains the initial conditions and is given by

$$\varphi(\boldsymbol{v},x,p)=f(\boldsymbol{v},x,0)\,(p+i\,k\,v_y) + v_x\frac{\partial f(\boldsymbol{v},x,0)}{\partial x}+\frac{\partial f(\boldsymbol{v},x,0)}{\partial t}. \tag{3.3}$$

Furthermore $\langle v_z\rangle_p=\int f_p\,v_z\,\mathrm{d}\boldsymbol{v}$. (3.4)

To specify the initial conditions we may assume that the plasma at $t=0$ is diplaced uniformly in the z-direction. With regard to the total distribution function at $t=0$ this means

$$F(\boldsymbol{v},\boldsymbol{r},0)=f_0(\boldsymbol{v}-\boldsymbol{v}_0). \tag{3.5}$$

The constant vector $\boldsymbol{v}_0$ will be directed along the z-axis since we assume that the plasma is displaced in this direction. The initial value of the disturbance is then obviously given by

$$f(\boldsymbol{v},x,0)=f_0(\boldsymbol{v}-\boldsymbol{v}_0)-f_0(\boldsymbol{v}). \tag{3.6}$$

Instead of the displaced velocity distribution as given by (3.5) we can, consistent with the assumption of infinitesimal wave amplitudes, use as the initial disturbance a power series expansion of (3.6) up to the first order:

$$f(\boldsymbol{v},x,0)=-\boldsymbol{v}_0\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}=-v_0\frac{\partial f_0}{\partial v_z}. \tag{3.7}$$

Although the initial distribution function given by (3.7) does not correspond to a uniformly displaced distribution function, it corresponds to a uniformly displaced plasma because we have for the expectation value of v_z:

$$\langle v_z\rangle=-v_0\int v_z\frac{\partial f_0}{\partial v_z}\,\mathrm{d}\boldsymbol{v}=-v_0\int_{-\infty}^{+\infty}v_z\frac{\mathrm{d}f_0}{\mathrm{d}v_z}\,\mathrm{d}v_z=v_0, \tag{3.8}$$

where for the distribution function $f_0(v)$ we make assumption

$$f_0(\boldsymbol{v})=f_0(v_x)\,f_0(v_y)\,f_0(v_z). \tag{3.9}$$

$f_0(v)$ shall be a symmetric function, $f_0(-v)=f_0(v)$, and shall go to zero more rapidly than $|v|^{-1}$ for $v\to\pm\infty$.

For convenience we use the normalization

$$\int_{-\infty}^{+\infty}f_0(v_x)\,\mathrm{d}v_x=\int_{-\infty}^{+\infty}f(v_y)\,\mathrm{d}v_y=\int_{-\infty}^{+\infty}f(v_z)\,\mathrm{d}v_z=1. \tag{3.10}$$

In computing (3.8) we made use of

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}f_0(v_z)}{\mathrm{d}v_z}v_z\,\mathrm{d}v_z=v_z f_0(v_z)\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}f_0(v_z)\,\mathrm{d}v_z=1. \tag{3.11}$$

Inserting (3.7) into (3.3) we obtain for the initial condition

$$\varphi=-v_0(p+i\,k\,v_y)\frac{\partial f_0}{\partial v_z}. \tag{3.12}$$

Inserting (3.12) into (3.2) we have after rearranging

$$\frac{\partial f_p}{\partial x}+\frac{p+i\,k\,v_y}{v_x}f_p-\frac{1}{p\,v_x}\frac{\partial f_0}{\partial v_z}\left[u^2k^2\langle v_z\rangle_p-v_0(p+i\,k\,v_y)\right]=0. \tag{3.13}$$

For the following we consider ALFVÉN waves with small wave numbers k and neglect for this reason the term $i\,k\,v_y$ against p. This is equivalent to removing ordinary LANDAU damping. In this approximation we have instead of (3.13)

$$\frac{\partial f_p}{\partial x}+\frac{p}{v_x}f_p-\frac{1}{p\,v_x}\frac{\partial f_0}{\partial v_z}\left[u^2k^2\langle v_z\rangle_p-v_0\,p\right]=0\,. \tag{3.14}$$

Eq. (3.14) can be considered as an ordinary linear differential equation of the first order for f_p as a function of x. Integration of (3.14) with regard to x leads to

$$f_p=\frac{\partial f_0}{\partial v_z}\frac{e^{-px/v_x}}{p\,v_x}\int(u^2k^2\langle v_z\rangle_p-v_0\,p)\,e^{px'/v_x}\,\mathrm{d}x'. \tag{3.15}$$

In order to determine in (3.15) the constant of integration we proceed as follows. We assume that $x>0$ and can make a similar consideration for $x<0$. Then for $v_x<0$ we have

$$\lim_{x\to\infty}e^{-px/v_x}\to\infty \tag{3.16}$$

The constant of integration in (3.15) is now determined in such a way to obtain a finite result. This leads to

$$f_p=-\frac{\partial f_0}{\partial v_z}\frac{e^{-px/v_x}}{p\,v_x}\int_x^\infty(u^2k^2\langle v_z\rangle_p-v_0\,p)\,e^{px'/v_x}\,\mathrm{d}x',\qquad v_x<0\,. \tag{3.17}$$

To obtain f_p for the case $v_x>0$ we have to consider a boundary condition to be imposed upon f_p. We shall for convenience assume that our magnetic field has to property

$$u^2(-x)=u^2(x)\,. \tag{3.18}$$

The natural boundary is thus the interface $x=0$ with regard to which $u^2(x)$ is symmetric. At $t=0$ the plasma is as a whole displaced uniformly in the z-direction with the consequence that only symmetric modes $\langle v_z\rangle\{-x\}=\langle v_z\rangle\{x\}$ are exited. If the solution is symmetric all particles behave at the interface $x=0$ as if they are reflected elastically from a rigid wall. The distribution function f_p must therefore obey at $x=0$ the boundary condition:

$$f_p(-v_x,0)=f_p(v_x,0)\,. \tag{3.19}$$

To obtain f_p for $v_x>0$ we must choose the constant of integration in such a way as to satisfy the boundary condition (3.19). The result is:

$$f_p=\frac{\partial f_0}{\partial v_x}\frac{e^{-px/v_x}}{p\,v_x}\left[\int_0^\infty(u^2k^2\langle v_z\rangle_p-v_0\,p)\,e^{-px'/v_x}\,\mathrm{d}x'+\int_0^x(u^2k^2\langle v_z\rangle_p-v_0\,p)\,e^{px'/v_x}\,\mathrm{d}x'\right],\qquad v_x>0\,. \tag{3.20}$$

In order to obtain f_p from (3.17) and (3.20) we have to know $\langle v_z\rangle_p$ which by itself is determined only after f_p is known. We therefore compute first $\langle v_z\rangle_p$ using the expression for f_p given by (3.17) and (3.20).

By definition we have

$$\langle v_z\rangle_p=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f_p(x,v_x,v_y,v_z)\,v_z\,\mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z\,. \tag{3.21}$$

We decompose $\langle v_z\rangle_p$ as follows:

$$\langle v_z\rangle_p=\langle v_z\rangle_p^{+}+\langle v_z\rangle_p^{-} \tag{3.22}$$

where

$$\langle v_z\rangle_p^{-}=\iint_{-\infty}^{+\infty}\mathrm{d}v_y\,\mathrm{d}v_z\int_{-0}^{\infty}\mathrm{d}v_x\,v_z\,f_p\,,\ v_x<0\,,\qquad \langle v_z\rangle_p^{+}=\iint_{-\infty}^{+\infty}\mathrm{d}v_y\,\mathrm{d}v_z\int_{0}^{\infty}\mathrm{d}v_x\,v_z\,f_p\,,\ v_x>0\,. \tag{3.23 a, b}$$

In order to compute $\langle v_z\rangle_p^{-}$ we have to use (3.17) and for $\langle v_z\rangle_p^{+}$ (3.20).

After the integrations over $\mathrm{d}v_y$ and $\mathrm{d}v_z$ have been carried out, we drop the index x on v_x and call $v_x\equiv v$. From (3.23 a) and (3.17) we thus obtain

$$\langle v_z\rangle_p^{-}=\frac{k^2}{p}\int_{-\infty}^{0}\mathrm{d}v\,\frac{f_0(v)}{v}\,e^{-px/v}\int_x^\infty\left(u^2\langle v_z\rangle_p-\frac{v_0\,p}{k^2}\right)e^{px'/v}\,\mathrm{d}x', \tag{3.24}$$

and from (3.23 b) inserting (3.20),

$$\langle v_z \rangle_p^+ = -\frac{k^2}{p}\left[\int_0^\infty \mathrm{d}v\, \frac{f_0(v)}{v}\, e^{-px/v} \int_0^\infty \left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right) e^{-px'/v}\, \mathrm{d}x' \right. \\ \left. + \int_0^\infty \mathrm{d}v\, \frac{f_0(v)}{v}\, e^{-px/v} \int_0^x \left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right) e^{px'/v}\, \mathrm{d}x' \right]. \tag{3.25}$$

In the integral for $\langle v_z \rangle_p^-$ [eq. (3.24)] we change the variable of integration v from $v \to -v$. Therefore $\mathrm{d}v \to -\mathrm{d}v$, $f_0(-v) = f_0(v)$. The result upon (3.24) is:

$$\langle v_z \rangle_p^- = \frac{k^2}{p} \int_\infty^0 \mathrm{d}v\, \frac{f_0(v)}{v}\, e^{px/v} \int_x^\infty \left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right) e^{-px'/v}\, \mathrm{d}x' \\ = -\frac{k^2}{p} \int_0^\infty \mathrm{d}v\, \frac{f_0(v)}{v}\, e^{px/v} \int_x^\infty \left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right) e^{-px'/v}\, \mathrm{d}x'. \tag{3.26}$$

Inserting the expressions (3.26) and (3.25) into (3.22) we obtain

$$-\langle v_z \rangle_p = \frac{k^2}{p}\left[\int_0^\infty \mathrm{d}v\, \frac{f_0(v)}{v}\, e^{-px/v} \int_0^\infty \left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right) e^{-px'/v}\, \mathrm{d}x' \right. \\ \left. + \int_0^\infty \mathrm{d}v\, \frac{f_0(v)}{v}\, e^{-px/v} \int_0^x \left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right) e^{px'/v}\, \mathrm{d}x' + \int_0^\infty \mathrm{d}v\, \frac{f_0(v)}{v}\, e^{px/v} \int_x^\infty \left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right) e^{-px'/v}\, \mathrm{d}x' \right]. \tag{3.27}$$

If we introduce the function

$$K(x) = \frac{k^2}{p} \int_0^\infty \mathrm{d}v\, \frac{f_0(v)}{v}\, e^{-px/v}, \tag{3.28}$$

eq. (3.27) can be written as follows

$$-\langle v_z \rangle_p = \int_0^\infty K(x+x')\left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right)\mathrm{d}x' + \int_0^x K(x-x')\left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right)\mathrm{d}x' + \int_x^\infty K(x'-x)\left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k}\right)\mathrm{d}x'. \tag{3.29}$$

A further simplification of (3.29) is possible by using (3.18) and the circumstance that only even modes are exited that is $\langle v_z \rangle_p\{-x\} = \langle v_z \rangle_p\{x\}$. The first integral occurring on the right hand side of eq. (3.29) can be transformed by substituting $x' \to -x'$:

$$\int_0^\infty K(x+x')\left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right)\mathrm{d}x' = -\int_0^{-\infty} K(x-x')\left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right)\mathrm{d}x'. \tag{3.30}$$

The second and third integral of the right hand side of eq. (3.29) can be combined by continuing $K(x)$ to negative arguments so that

$$K(-x) = K(x), \tag{3.31}$$

which means

$$K(x) = \frac{k^2}{p} \int_0^\infty \mathrm{d}v\, \frac{f_0(v)}{v}\, e^{-p|x|/v}. \tag{3.32}$$

We therefore can write

$$\int_0^x K(x-x')\left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right)\mathrm{d}x' + \int_x^\infty K(x'-x)\left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right)\mathrm{d}x' = \int_0^\infty K(|x-x'|)\left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right)\mathrm{d}x'. \tag{3.33}$$

Using (3.30) and (3.33) we can write for (3.29)

$$-\langle v_z \rangle_p = \int_{-\infty}^{+\infty} K(|x-x'|)\left(u^2 \langle v_z \rangle_p - \frac{v_0\, p}{k^2}\right)\mathrm{d}x' = \int_{-\infty}^{+\infty} K(|x-x'|)\; u^2 \langle v_z \rangle_p\, \mathrm{d}x' - \frac{v_0\, p}{k^2}\int_{-\infty}^{+\infty} K(|x-x'|)\, \mathrm{d}x'. \tag{3.34}$$

The second integral occurring on the right hand side of (3.34) can be evaluated by using the explicit form of $K(|x|)$ given by (3.32):

$$\int_{-\infty}^{+\infty} K(|x-x'|)\,dx' = \frac{k^2}{p}\int_{-\infty}^{+\infty} dx' \int_0^{\infty} dv\, \frac{f_0(v)}{v}\, e^{-p|x-x'|/v}$$
$$= 2\,\frac{k^2}{p}\int_0^{\infty} dx \int_0^{\infty} dv\, \frac{f_0(v)}{v}\, e^{-px/v} = 2\,\frac{k^2}{p^2}\int_0^{\infty} f_0(v)\,dv = \frac{k^2}{p^2}\int_{-\infty}^{+\infty} f_0(v)\,dv = \frac{k^2}{p^2}\,. \tag{3.35}$$

We thus obtain, finally, from (3.34)

$$-\langle v_z\rangle_p = \int_{-\infty}^{+\infty} K(|x-x'|)\,u^2 \langle v_z\rangle_p\,dx' - \frac{v_0}{p}\,. \tag{3.36}$$

4. Solution for a Zero Temperature Plasma

In the case of a zero temperature plasma, one should obtain the same results as those following from the magnetofluid dynamic equations. This can indeed be easily verified. For a zero temperature plasma we have for the undisturbed distribution function:

$$f_0(v) = \delta(v)\,. \tag{4.1}$$

We thus obtain for the kernel of the integral equation:

$$K(x) = \frac{k^2}{p}\int_0^{\infty} dv\, \frac{\delta(v)}{v}\, e^{-p|x|/v} = \frac{k^2}{2\,p}\lim_{v\to 0}\frac{e^{-p|x|/v}}{v} = \begin{cases} 0, & x>0 \\ \infty, & x=0\,. \end{cases} \tag{4.2}$$

Since

$$\int_{-\infty}^{+\infty} K(x)\,dx = \frac{k^2}{2\,p}\lim_{v\to 0}\frac{1}{v}\int_{-\infty}^{+\infty} e^{-p|x|/v}\,dx = \frac{p^2}{k^2}, \tag{4.3}$$

we have clearly $K(x) = (k^2/p^2)\,\delta(x)\,.$ (4.4)

Inserting (4.4) into (3.36) and solving for $\langle v_z\rangle_p$ results in

$$\langle v_z\rangle_p = v_0\,p/(p^2+k^2\,u^2)\,. \tag{4.5}$$

To obtain from (4.5) $\langle v_z\rangle$ we have to perform the inverse LAPLACE transform with the result:

$$\langle v_z\rangle = v_0 \cos[u(x)\,k\,t]\,. \tag{4.6}$$

Solution (4.6) is clearly identical with the magnetofluid dynamic solution for ALFVÉN waves with wave number k and amplitude v_0, propagating in an inhomogeneous magnetic field. The solution does not exhibit any damping, a result we would expect from a collisionless zero temperature plasma.

5. Constant Magnetic Field Gradient

For the sake of computational simplicity, we shall assume that the magnetic field, and thus the ALFVÉN velocity, has a linear dependence on x. That is

$$u(x) = |\nabla u|\,x \tag{5.1}$$

where $|\nabla u| = du/dx = \text{const}\,.$ (5.2)

With this assumption the integral eq. (3.36) has the form:

$$-\langle v_z\rangle_p = (\nabla u)^2 \int_{-\infty}^{+\infty} K(|x-x'|)\,x'^2 \langle v_z\rangle_p\,dx' - \frac{v_0}{p}\,. \tag{5.3}$$

6. Transformation to Nondimensional Variables

In order to solve the integral eq. (5.3) it is expedient to transform it to certain nondimensional variables. We introduce the DEBYE-length a and the ion plasma-frequency ω_0,

$$a^2 = k\,T/(4\,\pi\,n\,e^2)\,, \tag{6.1}$$

$$\omega_0{}^2 = 4\,\pi\,n\,e^2/m\,. \tag{6.2}$$

We then put

$$x = (a\,\omega_0/p)\,\xi\,,\quad v = a\,\omega_0\,\eta\,,\quad \langle v_z\rangle_p = (v_0/\omega_0)\,\varphi\,,$$
$$\xi = (p/a\,\omega_0)\,x,\quad \eta = (v/a\,\omega_0),\quad \varphi = (\omega_0/v_0)\,\langle v_z\rangle_p \tag{6.3}$$

where $a\,\omega_0 = \sqrt{k\,T/m}$ (6.4)

is the average thermal speed of the particles.

Using the explicit form for the kernel $K(x)$, the integral eq. (5.3) can be written as follows

$$-\langle v_z\rangle_p = \frac{k^2}{p}(\nabla u)^2 \int_0^\infty \mathrm{d}v \frac{f_0(v)}{v} \int_{-\infty}^{+\infty} \mathrm{d}x'\, e^{-p|x-x'|/v}\, x'^2 \langle v_z\rangle_p - \frac{v_0}{p}. \tag{6.5}$$

We assume furthermore, that the undisturbed distribution function $f_0(v)$ is given by a MAXWELLian being conveniently expressed by

$$f_0(v) = \frac{1}{\sqrt{2\pi}}\frac{1}{a\,\omega_0} \exp\left\{-\frac{v^2}{2(a\,\omega_0)^2}\right\}. \tag{6.6}$$

Eq. (6.5) thus takes the form

$$-\langle v_z\rangle_p = \frac{k^2}{\sqrt{2\pi}\,a\,\omega_0\,p}(\nabla u)^2 \int_0^\infty \mathrm{d}v \frac{\exp\left\{-\frac{v^2}{2(a\,\omega_0)^2}\right\}}{v} \int_{-\infty}^{+\infty} \mathrm{d}x'\, e^{-p|x-x'|/v}\, x'^2 \langle v_z\rangle_p - \frac{v_0}{p}. \tag{6.7}$$

Introducing the new variables defined by (6.3) eq. (6.7) can be brought to the nondimensional form

$$\varphi(\xi) = -\Lambda \int_0^\infty \mathrm{d}\eta \frac{e^{-\eta^2/2}}{\eta} \int_{-\infty}^{+\infty} \mathrm{d}\xi'\, e^{-|\xi-\xi'|/\eta}\, \xi'^2 \varphi(\xi') + \frac{\omega_0}{p}, \tag{6.8}$$

where we used the abbreviation $\Lambda = k^2(a\,\omega_0)^2(\nabla u)^2/[\sqrt{2\pi}\,p^4]$. (6.9)

7. Fourier-Transform of the Integral-Equation

We write eq. (6.8) as follows

$$\varphi(\xi) = -\Lambda \int_{-\infty}^{+\infty} K(|\xi-\xi'|)\, \xi'^2\, \varphi(\xi')\, \mathrm{d}\xi' + \omega_0/p \tag{7.1}$$

where

$$K(\xi) = \int_0^\infty \frac{e^{-\eta^2/2-|\xi|/\eta}}{\eta}\,\mathrm{d}\eta. \tag{7.2}$$

Applying a FOURIER transform on eq. (7.1) and making use of the faltung theorem results in

$$\varphi(\varkappa) = \sqrt{2\pi}\,\Lambda\, K(\varkappa) \frac{\mathrm{d}^2\varphi(\varkappa)}{\mathrm{d}\varkappa^2} + \sqrt{2\pi}\,\frac{\omega_0}{p}\,\delta(\varkappa). \tag{7.3}$$

In eq. (7.3) we have clearly

$$\varphi(\varkappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\xi)\, e^{-i\varkappa\xi}\,\mathrm{d}\xi,$$

$$K(\varkappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} K(\xi)\, e^{-i\varkappa\xi}\,\mathrm{d}\xi. \tag{7.4}$$

From eq. (7.2) we obtain for the FOURIER transform of $K(\varkappa)$

$$K(\varkappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\xi\, e^{-i\varkappa\xi} \int_0^\infty \mathrm{d}\eta \frac{e^{-\eta^2/2-|\xi|/\eta}}{\eta}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}\eta \frac{e^{-\eta^2/2}}{\eta} \int_{-\infty}^{+\infty} \mathrm{d}\xi\, e^{-i\varkappa\xi-|\xi|/\eta}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{d}\eta \frac{e^{-\eta^2/2}}{\eta} \int_0^\infty \mathrm{d}\xi \cos(\varkappa\,\xi)\, e^{-\xi/\eta}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-\eta^2/2}}{1+\varkappa^2\eta^2}\,\mathrm{d}\eta, \tag{7.5}$$

and making use of the formula

$$\int_0^\infty \frac{e^{-x^2y^2}}{1+y^2}\,\mathrm{d}y = \sqrt{\pi}\, e^{x^2} \int_x^\infty e^{-z^2}\,\mathrm{d}z = \frac{\pi}{2}\, e^{x^2}[1-\Phi(x)], \tag{7.6}$$

where Φ is the normalized error integral, we can write

$$K(\varkappa) = \sqrt{\frac{\pi}{2}}\, \frac{e^{1/2\varkappa^2}}{\varkappa}\,[1-\Phi(1/\sqrt{2}\,\varkappa)]. \tag{7.7}$$

8. W.K.B. Solution of the Differential Equation for $\varphi(\varkappa)$

We start from eq. (7.3) which we integrate over the delta function from $\varkappa=-\varepsilon$ to $\varkappa=+\varepsilon$, $\varepsilon \ll 1$. Using the value $K(0)=1$ we thus obtain

$$\Lambda[\varphi'(\varepsilon)-\varphi'(-\varepsilon)] + \omega_0/p = 0. \tag{8.1}$$

$\varphi(\varkappa)$ is a symmetric function of $\varkappa$ since $\varphi(\xi)$ was assumed to be a symmetric function of ξ. We, therefore, must have

$$\varphi'(-\varepsilon) = \varphi'(\varepsilon). \tag{8.2}$$

At $\varkappa = 0$ the derivative jumps and we have

$$\lim_{\varkappa \to 0} \varphi'(\varkappa) = \pm \omega_0 / 2 \Lambda p, \quad \varkappa \lesseqgtr 0. \tag{8.3}$$

Because of the symmetry with respect to $\varkappa$ it is sufficient to restrict the computation of $\varphi(\varkappa)$ to values $\varkappa > 0$ while using the boundary condition

$$\varphi'(0) = -\omega_0 / 2 \Lambda p . \tag{8.4}$$

For $\varkappa > 0$ we may write eq. (7.3) in a different form

$$\varphi''(\varkappa) - E^2 V(\varkappa) \varphi(\varkappa) = 0 , \tag{8.5}$$

with $$E^2 = \frac{1}{\sqrt{2\pi} \Lambda} = \frac{p^4}{(k a)^2 \omega_0^2 (\nabla u)^2}, \tag{8.6}$$

and $$V(\varkappa) = K^{-1}(\varkappa) = \sqrt{\frac{2}{\pi}} \frac{\varkappa e^{-1/2 \varkappa^2}}{[1 - \Phi(1/\sqrt{2} \varkappa)]} . \tag{8.7}$$

Eq. (8.5) has the W.K.B. solution

$$\varphi(\varkappa) = \frac{A}{\sqrt[4]{V(\varkappa)}} \exp\left[-E \int_0^\varkappa \sqrt{V(\varkappa)} \, d\varkappa \right], \tag{8.8}$$

where the constant A must be determined from the boundary condition (8.4). If $\varkappa \ll 1$ we have for (8.7) the asymptotic expansion

$$V(\varkappa) = \frac{1}{1 - \varkappa^2 + 3 \varkappa^4 - 15 \varkappa^6 + ..}, \quad \varkappa \ll 1 . \tag{8.9}$$

In the neighborhood of $\varkappa = 0$, with $\varkappa > 0$, $\varphi(\varkappa)$ according to (8.8) can be approximated by

$$\varphi(\varkappa) = A e^{-E\varkappa} \tag{8.10}$$

and therefore $$\varphi'(0) = -A E . \tag{8.11}$$

From (8.4) and (8.11) we thus obtain

$$A = \frac{\omega_0}{2 E \Lambda p} = \sqrt{\frac{\pi}{2}} \frac{\omega_0 E}{p} = \sqrt{\frac{\pi}{2}} \frac{p}{k a |\nabla u|} . \tag{8.12}$$

We use the following abbreviation:

$$A = c p , \quad c = \sqrt{\frac{\pi}{2}} \frac{1}{k a |\nabla u|} , \tag{8.13}$$

and arrive at

$$\varphi(\varkappa) = \frac{c p}{\sqrt[4]{V(\varkappa)}} \exp\left[-E \int_0^\varkappa \sqrt{V(\varkappa)} \, d\varkappa \right]. \tag{8.14}$$

In order to determine $\varphi(\varkappa)$ for arbitrary values of $\varkappa$ we have to compute the integral under the exponential of (8.14). This integral has been evaluated numerically and the result of this integration is shown in the Fig. 1, where a plot of

$$\int_0^\varkappa \sqrt{V(\varkappa)} \, d\varkappa \quad \text{and} \quad \int_0^\varkappa \sqrt{V(\varkappa)} \, d\varkappa - \varkappa \quad \text{is given.}$$

One can easily show that within an error of a few precent over a range from $\varkappa = 0$ to $\varkappa = 10$ this function can be approximated by

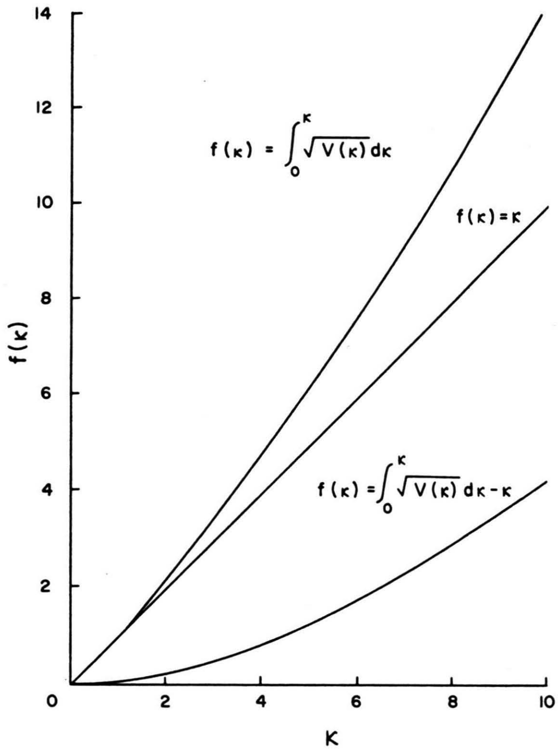

Fig. 1. The function $\int_0^\varkappa \sqrt{V(\varkappa)} \, d\varkappa$ and $\int_0^\varkappa \sqrt{V(\varkappa)} \, d\varkappa - \varkappa$.

$$\int_0^\varkappa \sqrt{V(\varkappa)} \, d\varkappa \cong \varkappa + 0.043 \varkappa^2 . \tag{8.15}$$

From this follows also

$$\sqrt{V(\varkappa)} \cong 1 + 0.086 \varkappa . \tag{8.16}$$

We, therefore, may put with sufficient accuracy in (8.14) $\sqrt[4]{V(\varkappa)} \cong 1$. We thus finally obtain with very good accuracy for the W.K.B. approximation of $\varphi(\varkappa)$ the expression

$$\varphi(\varkappa) \cong c p e^{-E(\varkappa + 0.043 \varkappa^2)} . \tag{8.17}$$

9. The Inverse Fourier and Laplace Transforms

Using the FOURIER transform as given by eq. (8.17) the inverse transforms can be carried out in a closed form. The inverse FOURIER transform is given by

$$\varphi(\xi) = c p \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(\varkappa \xi) e^{-E\varkappa - \gamma E \varkappa^2} d\varkappa , \tag{9.1}$$

where $$\gamma = 0.043 . \tag{9.2}$$

The integral (9.1) can be carried out in a closed form:

$$\varphi(\xi) = \frac{c\,p}{2\sqrt{2}\,\gamma\,E}\left\{\exp\left[\frac{(E-i\,\xi)^2}{4\,\gamma\,E}\right]\cdot\text{Erfc}\left[\frac{E-i\,\xi}{2\sqrt{\gamma\,E}}\right]+\exp\left[\frac{(E+i\,\xi)^2}{4\,\gamma\,E}\right]\cdot\text{Erfc}\left[\frac{E+i\,\xi}{2\sqrt{\gamma\,E}}\right]\right\}. \tag{9.3}$$

In eq. (9.3) $\text{Erfc} = 1 - \Phi$, where Φ is the normalized error integral.

Next we use the abbreviations

$$E = \alpha\,p^2, \qquad \xi = \beta\,p\,x, \qquad p_0 = i(\beta/\alpha)\,x = i\,k\,|\nabla u|\,x,$$
$$\alpha = 1/(k\,a\,\omega_0\,|\nabla u|), \qquad \beta = 1/a\,\omega_0. \tag{9.4}$$

From (6.3) we know that $\varphi = (\omega_0/v_0)\,\langle v_z\rangle_p$, thus we may write for (9.3):

$$\langle v_z\rangle_p = \frac{v_0\,c}{2\sqrt{2}\,\alpha\,\gamma\,\omega_0}\left\{\exp\left[\frac{\alpha}{4\,\gamma}\,(p-p_0)^2\right]\cdot\text{Erfc}\left[\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\,(p-p_0)\right]\right.$$
$$\left.+\exp\left[\frac{\alpha}{4\,\gamma}\,(p+p_0)^2\right]\cdot\text{Erfc}\left[\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\,(p+p_0)\right]\right\}. \tag{9.5}$$

In the LAPLACE inversion formula

$$\langle v_z\rangle = \frac{1}{2\,\pi\,i}\int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon}\langle v_z\rangle_p\,e^{pt}\,dp, \tag{9.6}$$

we make the substitution of a new variable p^*, $\quad p \mp p_0 = p^*$, (9.7)

where the two signs are to be taken for the integrals over the first and second term occurring in the curly bracket on the right hand side of (9.5).

We obtain

$$\langle v_z\rangle = \frac{v_0\,c}{2\sqrt{2}\,\alpha\,\gamma\,\omega_0}\,\{e^{p_0 t}+e^{-p_0 t}\}\cdot\frac{1}{2\,\pi\,i}\int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon}\exp\{(\alpha/4\,\gamma)\,p^{*2}\}\cdot\text{Erfc}\left[\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\,p^*\right]e^{p^* t}\,dp^*. \tag{9.8}$$

We then use the formula

$$\frac{1}{2\,\pi\,i}\int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon}e^{ap^2}\,\text{Erfc}(\sqrt{a}\,p)\,e^{pt}\,dp = \pi^{-1/2}\,a^{-1/2}\,e^{-t^2/4a}, \tag{9.9}$$

and obtain finally from (9.8), after inserting the values for α, β and γ:

$$\langle v_z\rangle = v_0\cos(k\,|\nabla u|\,x\,t)\,\exp\{-0.043\,k\,a\,\omega_0\,|\nabla u|\,t^2\}, \tag{9.10}$$

or

$$\langle v_z\rangle = v_0\cos(k\,|\nabla u|\,x\,t)\,e^{-(t/\tau)^2}, \tag{9.11}$$

where we introduced a damping time τ given by $\quad \tau = 4.82\,(k\,a\,\omega_0\,|\nabla u|)^{-1/2}$. (9.12)

10. The Distribution Function

Finally we would like to know how the distribution function changes with time. Knowing $\langle v_z\rangle_p$ the distribution function can be computed from (3.17) for $v_x<0$ and from (3.20) for $v_x>0$.

As a first approximation we may insert for $\langle v_z\rangle_p$ the expression given by (4.5) which is valid for a zero temperature plasma. For a warm plasma we have to use for $\langle v_z\rangle_p$ expression (9.5). One should expect, however, that the largest contribution to $f(x,t)$ in a warm plasma results from inserting the value $\langle v_z\rangle_p$ valid for a cold plasma. Inserting the value $\langle v_z\rangle_p$ given by (4.5) into eqs. (3.17) and (3.20) we have

$$f_p = v_0\,\frac{\partial f_0}{\partial v_z}\,\frac{p^2}{v_x}\int_x^{\infty}\frac{\exp\{p(x'-x)/v_x\}}{p^2+k^2\,u^2(x')}\,dx', \quad v_x<0, \tag{10.1}$$

$$f_p = -v_0\,\frac{\partial f_0}{\partial v_z}\,\frac{p^2}{v_x}\left[\int_0^{\infty}\frac{\exp\{-p(x+x')/v_x\}}{p^2+k^2\,u^2(x')}\,dx'+\int_0^{x}\frac{\exp\{-p(x-x')/v_x\}}{p^2+k^2\,u^2(x')}\,dx'\right], \quad v_x>0. \tag{10.2}$$

Using the inversion formula
$$f(x, t) = \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon} f_p\, e^{pt}\, \mathrm{d}p \tag{10.3}$$
and writing $u^2(x') = |\nabla u|^2 x'^2$ we obtain
$$f(x, t) = -v_0 \frac{\partial f_0}{\partial v_z} \frac{k|\nabla u|}{v_x} \int_x^\infty x' \sin\left[k|\nabla u|x'\left(t + \frac{x'-x}{v_x}\right)\right] \mathrm{d}x', \quad v_x < 0, \tag{10.4}$$
$$f(x, t) = v_0 \frac{\partial f_0}{\partial v_z} \frac{k|\nabla u|}{v_x}\left[\int_0^\infty x' \sin\left[k|\nabla u|x'\left(t - \frac{x+x'}{v_x}\right)\right] \mathrm{d}x' + \int_0^x x' \sin\left[k|\nabla u|x'\left(t - \frac{x-x'}{v_x}\right)\right] \mathrm{d}x'\right], \quad v_x > 0. \tag{10.5}$$

To evaluate the integrals occurring in (10.4) and (10.5) we apply the saddlepoint method. For the integral in (10.4) we can write
$$\int_x^\infty x' \sin\left[k|\nabla u|x'\left(t + \frac{x'-x}{v_x}\right)\right] \mathrm{d}x' = \mathrm{Im}\int_x^\infty x' \exp\left[ik|\nabla u|x'\left(t + \frac{x'-x}{v_x}\right)\right] \mathrm{d}x' = \mathrm{Im}\int_x^\infty x'\, e^{H(x')}\, \mathrm{d}x', \tag{10.6}$$
where
$$H(x') = ik|\nabla u|\left[\frac{1}{v_x} x'^2 + \left(t - \frac{x}{v_x}\right) x'\right]. \tag{10.7}$$
$H(x')$ as a function of x' has a saddlepoint at
$$x_0' = \tfrac{1}{2}(x - v_x t). \tag{10.8}$$
Expanding around this saddlepoint we have
$$H(x') \simeq H(x_0') + \frac{1}{2}\left(\frac{\partial^2 H}{\partial x'^2}\right)_{x'=x_0'} \cdot (\Delta x)^2 = ik|\nabla u|\left[-\frac{(x - v_x t)^2}{4 v_x} + \frac{1}{v_x}(\Delta x)^2\right], \quad \Delta x = x' - x_0'. \tag{10.9}$$
From (10.8) it follows that for large values of t, $x_0' \gg x$ because $v_x < 0$. We can, therefore, extend the lower limit of integration to $-\infty$.

For the integral (10.6) we thus write
$$\mathrm{Im}\int_x^\infty x'\, e^{H(x')}\, \mathrm{d}x' \cong \mathrm{Im}\int_{-\infty}^{+\infty} x'\, e^{H(x')}\, \mathrm{d}x$$
$$\simeq \frac{1}{2}(x - v_x t)\, \mathrm{Im} \exp\left\{\frac{ik|\nabla u|}{4} \frac{(x - v_x t)^2}{-v_x}\right\} \int_{-\infty}^{+\infty} \exp\left\{-\frac{ik|\nabla u|}{-v_x}(\Delta x)^2\right\} \mathrm{d}(\Delta x). \tag{10.10}$$
In writing down (10.10) we took into account that $v_x < 0$; putting
$$x = \sqrt{\frac{-v_x}{k|\nabla u|}}\, e^{-\pi i/4} s, \tag{10.11}$$
we have
$$\int_{-\infty}^{+\infty} \exp\left\{-\frac{ik|\nabla u|}{-v_x}(\Delta x)^2\right\} \mathrm{d}(\Delta x) = \sqrt{\frac{-v_x}{|\nabla u|}}\, e^{-\pi i/4} \int_{-\infty}^{+\infty} e^{-s^2}\, \mathrm{d}s = \sqrt{\frac{-v_x \pi}{k|\nabla u|}}\, e^{-\pi i/4}. \tag{10.12}$$
Inserting (10.12) into (10.10) and taking the imaginary part results in
$$\mathrm{Im}\int_x^\infty x'\, e^{H(x')}\, \mathrm{d}x' \cong \frac{\sqrt{\pi}}{2}\sqrt{\frac{-v_x}{k|\nabla u|}}(x - v_x t) \sin\left[\frac{k|\nabla u|}{4} \frac{(x - v_x t)^2}{-v_x} - \frac{\pi}{4}\right]. \tag{10.13}$$
For the two integrals occurring in (10.5) we have
$$\int_0^\infty x' \sin\left[k|\nabla u|x'\left(t - \frac{x'+x}{v_x}\right)\right] \mathrm{d}x' = \mathrm{Im}\int_0^\infty x'\, e^{H_1(x')}\, \mathrm{d}x', \tag{10.14}$$
where
$$H_1(x') = ik|\nabla u|\left[-\frac{1}{v_x} x'^2 + \left(t - \frac{x}{v_x}\right) x'\right] \tag{10.15}$$

and
$$\int_0^x x' \sin\left[k\,|\nabla u|\,x'\left(t-\frac{x-x'}{v_x}\right)\right] \mathrm{d}x' = \mathrm{Im}\int_0^x x'\, e^{H_2(x')}\,\mathrm{d}x', \tag{10.16}$$

with
$$H_2(x') = i\,k\,|\nabla u|\left[\frac{1}{v_x}x'^2 + \left(t-\frac{x}{v_x}\right)x'\right]. \tag{10.17}$$

$H_1(x')$ has a saddlepoint at
$$x_0' = -\tfrac{1}{2}(x - v_x t)\,. \tag{10.18}$$

For large values of t this saddlepoint is at large values x_0'. The lower limit of the integral with $H_1(x')$ in the exponent can, therefore, be extended to $-\infty$. Expanding $H_1(x')$ around this saddlepoint we have

$$H_1(x') \cong H_1(x_0') + \frac{1}{2}\left(\frac{\partial^2 H}{\partial x'^2}\right)_{x'=x_0'} \cdot (\Delta x)^2 = i\,k\,|\nabla u|\left[\frac{(x-v_x t)^2}{4\,v_x} - \frac{1}{v_x}(\Delta x)^2\right], \quad \Delta x = x' - x_0'. \tag{10.19}$$

For the integral (10.14) we then have

$$\mathrm{Im}\int_0^\infty x'\,e^{H_1(x')}\,\mathrm{d}x' \cong -\frac{1}{2}(x - v_x t)\,\mathrm{Im}\exp\left\{\frac{i\,k\,|\nabla u|}{4}\,\frac{(x-v_x t)^2}{v_x}\right\}\int_{-\infty}^{+\infty}\exp\left\{-\frac{i\,k\,|\nabla u|}{v_x}(\Delta x)^2\right\}\mathrm{d}(\Delta x)$$
$$= -\frac{\sqrt{\pi}}{2}\sqrt{\frac{v_x}{k\,|\nabla u|}}\,(x - v_x t)\sin\left[\frac{k\,|\nabla u|}{4}\,\frac{(x-v_x t)^2}{v_x} - \frac{\pi}{4}\right]. \tag{10.20}$$

To compute the integral (10.16) we ask for the saddlepoint of $H_2(x')$ which is located at

$$x_0' = \tfrac{1}{2}(x - v_x t). \tag{10.21}$$

For large values of t the saddlepoint is located on the negative side of the x' axis, far away from the origin. Since the integral (10.16) is extended from $x'=0$ to $x'=x$, the contribution of this saddlepoint to the integral is very small. The value of the integral (10.16) is, therefore, small if compared with the value of the integral (10.14) and shall be neglected. Inserting the value of the integrals (10.6) and (10.14) into eqs. (10.4) and (10.5) we can write for the distribution function $f(x,t)$ an expression which is valid for $v_x \lessgtr 0$ in the form

$$f(x,t) = -\frac{\sqrt{\pi}}{2}\,v_0\,\frac{\partial f_0}{\partial v_z}\sqrt{\frac{k\,|\nabla u|}{|v_x|}}\,(x - v_x t)\sin\left[\frac{k\,|\nabla u|}{4}\,\frac{(x-v_x t)^2}{v_x} - \frac{\pi}{4}\right]. \tag{10.22}$$

To supplement these considerations we would like to show that even with the improved expression for $\langle v_z\rangle_p$ given by eq. (9.5) it is possible to carry out the inverse LAPLACE transform. The results are expressions for the distribution function given by definite integrals useful for numerical computation, if one should be interested to explore the behavior of the distribution function into further details.

We first consider the case $v_x<0$ and write for f_p according to (3.17)

$$f_p = -\frac{\partial f_0}{\partial v_z}\,\frac{k^2\,|\nabla u|^2}{v_x}\int_x^\infty x'^2\langle v_z\rangle_p\,\frac{\exp[p(x'-x)/v_x]}{p}\,\mathrm{d}x' - v_0\,\frac{\partial f_0}{\partial v_z}\,\frac{1}{p}\,. \tag{10.23}$$

Inserting into (10.23) expression (9.5) for the LAPLACE transform $\langle v_z\rangle_p$ we have

$$f_p = -\frac{\partial f_0}{\partial v_z}\,\frac{v_0\,c\,k^2\,|\nabla u|^2}{2\sqrt{2}\,a\,\gamma\,\omega_0\,v_x}\int_x^\infty x'^2\,\frac{\exp[p(x'-x)/v_x]}{p}\left\{\exp\left[\frac{a}{4\gamma}(p-p_0)^2\right]\cdot\mathrm{Erfc}\left[\frac{1}{2}\sqrt{\frac{a}{\gamma}}\,(p-p_0)\right]\right.$$
$$\left.+\exp\left[\frac{a}{4\gamma}(p+p_0)^2\right]\cdot\mathrm{Erfc}\left[\frac{1}{2}\sqrt{\frac{a}{\gamma}}\,(p+p_0)\right]\right\}\mathrm{d}x' - v_0\,\frac{\partial f_0}{\partial v_z}\,\frac{1}{p}\,. \tag{10.24}$$

In carrying out the inverse transform
$$f(x,t) = \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon} f_p\,e^{pt}\,\mathrm{d}p\,, \tag{10.25}$$

the last term in (10.24) resulting from the initial condition has the constant value $-v_0\cdot\partial f_0/\partial v_z$.

To carry out the integration over p in the integrals resulting from the first and second term in the curly bracket of (10.24) we make the substitution of a new variable p^* according to

$$p \mp p_0 = p^*. \tag{10.26)}$$

Inserting this substitution into (10.24) we obtain for the first and second integral:

$$\frac{\exp\{\pm p_0[t+(x'-x)/v_x]\}+i}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\exp\left[\frac{a}{4\gamma}p^{*2}\right]\cdot\mathrm{Erfc}\left[\frac{1}{2}\sqrt{\frac{a}{\gamma}}p^*\right]}{p^*\pm p_0}\exp\left\{p^*\left[t+\frac{x'-x}{v_x}\right]\right\}dp^*. \tag{10.27}$$

To evaluate these integrals we use the formula

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{p^2}\,\mathrm{Erfc}(p)}{p-a}e^{pt}\,dp=e^{a(t+a)}\left[\Phi(\tfrac{1}{2}t+a)-\Phi(a)\right]. \tag{10.28}$$

The result upon (10.27) after taking the sum of both integrals is given by

$$\exp\left[\frac{a}{4\gamma}p_0^2\right]\left\{\Phi\left[2\sqrt{\frac{\gamma}{a}}\left(t+\frac{x'-x}{v_x}\right)-\frac{1}{2}\sqrt{\frac{a}{\gamma}}p_0\right]+\Phi\left[2\sqrt{\frac{\gamma}{a}}\left(t+\frac{x'-x}{v_x}\right)+\frac{1}{2}\sqrt{\frac{a}{\gamma}}p_0\right]\right\}. \tag{10.29}$$

Since $p_0=i\,k\,|\nabla u|\,x'$ the arguments in the two error-integrals of (10.29) are complex conjugated. One easily verifies however, that their sum is real. Using the result given by (10.29) and inserting the expressions for α and β we can write for the distribution function

$$f(x,t)=-v_0\frac{\partial f_0}{\partial v_z}\left\{\frac{\sqrt{\pi}}{4\sqrt{\gamma}}\frac{k^{3/2}|\nabla u|^{3/2}}{(a\,\omega_0)^{1/2}v_x}\cdot\int_x^\infty x'^2\exp\left[-\frac{k|\nabla u|}{4\gamma a\omega_0}x'^2\right]\left\{\Phi\left[\frac{1}{2}\sqrt{\frac{k|\nabla u|}{\gamma a\omega_0}}\left(4\gamma a\omega_0\left(t+\frac{x'-x}{v_x}\right)-ix'\right)\right]\right.\right.$$

$$\left.\left.+\Phi\left[\frac{1}{2}\sqrt{\frac{k|\nabla u|}{\gamma a\omega_0}}\left(4\gamma a\omega_0\left(t+\frac{x'-x}{v_x}\right)+ix'\right)\right]\right\}dx'+1\right\},\quad v_x<0. \tag{10.30}$$

In a quite analogous way we obtain for positive velocities $v_x>0$:

$$f(x,t)=v_0\frac{\partial f_0}{\partial v_z}\left\{\frac{\sqrt{\pi}}{4\sqrt{\gamma}}\frac{k^{3/2}|\nabla u|^{3/2}}{(a\,\omega_0)^{1/2}v_x}\left[\int_0^\infty x'^2\exp\left[-\frac{k|\nabla u|}{4\gamma a\omega_0}x'^2\right]\right.\right.$$

$$\cdot\left\{\Phi\left[\frac{1}{2}\sqrt{\frac{k|\nabla u|}{\gamma a\omega_0}}\left(4\gamma a\omega_0\left(t-\frac{x'+x}{v_x}\right)-ix'\right)\right]+\Phi\left[\frac{1}{2}\sqrt{\frac{k|\nabla u|}{\gamma a\omega_0}}\left(4\gamma a\omega_0\left(t-\frac{x'+x}{v_x}\right)+ix'\right)\right]\right\}dx'$$

$$+\int_0^x x'^2\exp\left[-\frac{k|\nabla u|}{4\gamma a\omega_0}x'^2\right]\left\{\Phi\left[\frac{1}{2}\sqrt{\frac{k|\nabla u|}{\gamma a\omega_0}}\left(4\gamma a\omega_0\left(t+\frac{x'-x}{v_x}\right)-ix'\right)\right]\right. \tag{10.31}$$

$$\left.\left.\left.+\Phi\left[\frac{1}{2}\sqrt{\frac{k|\nabla u|}{\gamma a\omega_0}}\left(4\gamma a\omega_0\left(t+\frac{x'-x}{v_x}\right)+ix'\right)\right]\right\}dx'\right]-1\right\},\quad v_x>0.$$

11. Discussion of the Results

The most important result of this investigation is contained in eqs. (9.10) and (9.11), where we obtained for the damping time a value given by

$$\tau=4.82\,(k\,a\,\omega_0\,|\nabla u|)^{-1/2}. \tag{11.1}$$

Introducing into (11.1) the temperature according to

$$(a\,\omega_0)^2=k\,T/m \tag{11.2}$$

and $$|\nabla u|=\left|\nabla\left(\frac{H}{\sqrt{4\pi\varrho}}\right)\right|=\frac{1}{2\sqrt{\pi m}}\left|\nabla\left(\frac{H}{\sqrt{N}}\right)\right|, \tag{11.3}$$

(k is the Boltzmann constant, N the atomic number density), and finally the Alfvén wavelength $\lambda=2\pi/k$, we have

$$\tau=4.37\times10^{-8}\frac{A^{1/2}N^{1/4}T^{-1/4}\lambda^{1/2}}{[\sqrt{N}\,|\nabla_\perp(H/\sqrt{N})|]^{1/2}}\,[\mathrm{sec}]. \tag{11.4}$$

We used the symbol $\nabla_\perp$ to indicate that the gradient must be perpendicular to $H/\sqrt{N}$. From (11.4) it follows that the anomalous damping time τ depends not very strongly on the plasma density or plasma temperature. Interesting, however, is the dependence of τ upon the number density N and the temperature T. From (11.4) it follows that with decreasing N and increasing T, τ decreases. This behavior is opposite to the dependence on a collisional damping time upon N which decreases with increasing N and decreasing T.

We compare (11.4) with the ion collision time $t_c(Z=1)$,

$$t_c = 0.76\, A^{1/2}\, T^{3/2}/N \ \text{[sec]}, \tag{11.5}$$

taking the following numerical example:

$$T = 10^8\ ^\circ\text{K}, \quad N = 10^{15}\ \text{cm}^{-3}, \quad A = 2, \quad \lambda = 10^2\ \text{cm},$$
$$\sqrt{N}\,|\nabla_\perp (H/\sqrt{N})| = 10^3\ \text{gauss/cm},$$

and obtain

$$\tau = 1.1 \times 10^{-6}\ \text{sec}, \qquad t_c = 1.1 \times 10^{-3}\ \text{sec}.$$

It follows that for the given example the anomalous damping time is three orders of magnitude less than the collision time. Taking, however, $T = 10^6\ ^\circ$K we obtain $\tau = 3.5 \times 10^{-6}$ sec and $t_c = 1.1 \times 10^{-6}$ sec.

These results demonstrate that above temperatures of $10^6\ ^\circ$K the anomalous damping can become an important process. The very short damping times computed from (11.4) indicate that this damping may be useful to heat a plasma up to high temperatures in a time which is short compared with the collision time.

We would like to add one remark about the behavior of the distribution function at large values of time. Taking the form of $f(x, t)$ given by eq. (10.22) we can conclude that for large values of t,

$$f(x, t) \propto \exp\left\{i\ \frac{k\,|\nabla u|\,t^2}{4}\ v_x\right\}. \tag{11.6}$$

Introducing a wave number in velocity space which has the dimension (velocity)$^{-1}$ an which is given by $k_v = k\,|\nabla u|\,t^2/4$ we can write instead of (11.6) (putting $v_x \equiv v$):

$$f(x, t) \propto e^{ik_v v}. \tag{11.7}$$

The distribution function (11.7) has an oscillatory behavior in velocity space. With increasing time the wave number in velocity space increases proportional to t^2 with the constant of proportionality increasing with increasing $|\nabla u|$. As a result the distribution function becomes more and more oscillatory as time passes.

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